Potential theory and dimension

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Let $K_\alpha(x) = \begin{cases} |x|^{-\alpha}, & \alpha > 0 \\ -\log|x|, & \alpha = 0 \end{cases}$. Let μ be a Borel measure in $\mathbb{R}^d$. The α-potential of μ is defined as

$$U_\alpha^\mu(x) := \int_{\mathbb{R}^d} K_\alpha(x-y)\, d\mu(y).$$

Special role is played by $\alpha = d-2$. In this case, $\Delta K_{d-2}(x) = \sigma_d \delta_0$, for some σ_d depending on d $\left(\sigma_d = \begin{cases} -2\pi, & d=2 \\ -(d-2)\,Area(S^d) \end{cases}\right)$.

so $\Delta U_\alpha^\mu = \sigma_d \mu$ – solution to a Laplace problem. In this case, U_μ^α is harmonic outside supp μ. For $\alpha \geq d-2$ – subharmonic.

The α-energy of μ:

$$I_\alpha(\mu) := \iint K_\alpha(x-y)\, d\mu(x)\, d\mu(y) = \int U_\alpha^\mu(x)\, d\mu(x).$$

Familiar electrostatic energy for $\alpha = 1$, $d = 3$.

The integral always exists, for compactly supported μ, since $K_\alpha(x-y)$ is bounded below on supp μ. Can be ∞.

Let E be a Borel set. The α-equilibrium constant of E is defined as $V_\alpha := \inf_{\operatorname{supp}\mu \subset E,\ \mu(E)=1} I_\alpha(\mu)$. $V_\alpha = \infty \iff \forall \mu : \operatorname{supp}\mu \subset E$, $I_\alpha(\mu) = \infty$.

α-capacity of E is defined as

$$Cap_\alpha(E) = \begin{cases} e^{-V_\alpha}, & \alpha = 0 \\ V_\alpha^{-1/\alpha}, & \alpha > 0 \end{cases} \qquad Cap_\alpha E > 0 \iff \exists \mu : \operatorname{supp}\mu \subset E,\ I_\alpha(\mu) < \infty$$

defined this way so that $cap_\alpha rE = r\, cap_\alpha E$.

Will concentrate on relations between capacity and dimension.

Thm (Frostman) Let E – compact, $H_h(E) > 0$ for some h with $\int_0 \frac{h(t)}{t^{1+\alpha}}\, dt < \infty$. Then $Cap_\alpha(E) > 0$.

In particular, if $Cap_\alpha(E) = 0$, then for any $\beta > \alpha$, $H_\beta(E) = 0$, and thus $H\dim E \leq \alpha$.

Proof. By Frostman's lemma, $\exists$ h-smooth μ.

Let us show that $I_\alpha(\mu) < \infty$ by showing that $\|U_\alpha^\mu\|_\infty < \infty$.

Let $n(t) := \mu(B(x,t)) \leq C h(t)$. $R := \operatorname{diam} E$.

Then (at least for $\alpha > 0$):

$$V_\alpha(x) = \int_0^R t^{-\alpha}\, dn(t) = \lim_{\varepsilon \to 0}\left(\left(\frac{n(t)}{t^\alpha}\right)\Bigg|_\varepsilon^R + \alpha \int_\varepsilon^R \frac{n(t)\,dt}{t^{\alpha+1}}\right) \leq$$

$$\frac{n(R)}{R^\alpha} + \alpha \int_0^R \frac{n(t)\,dt}{t^{\alpha+1}} \leq \frac{h(R)}{R^\alpha} + C\alpha \int_0^\infty \frac{h(t)\,dt}{t^{\alpha+1}} \quad \blacksquare$$

Same for $\alpha = 0$.

For the other direction, we'll need lemma:

Lemma on singularity Let E be compact in $\mathbb{R}^d$, $m_h(E) < \infty$ for some gauge h, and $E_1 = \{x \in E : \lim \frac{\mu(Q_n(x))}{h(b^{-n})} = 0\}$, where $Q_n(x)$ is b^{-n}-b-adic cube containing x. Then $\mu(E_1) = 0$.

Proof. Let $\varphi_n(x) := \frac{\mu(Q_n(x))}{h(b^{-n})}$. Then on E_1 $\varphi_n \to 0$.

By Egoroff's thm, if $\mu(E_1) > 0$, $\exists$ $E_2 \subset E_1$, $\mu(E_2) > 0$ and $\varphi_n \to 0$ uniformly on E_2. Take n so that $\varphi_n(x) < \varepsilon$ $\forall x \in E_2$.

One can find a cover of E by b-cubes with sides $< b^{-n}$, and $\sum h(\operatorname{diam} Q_k) \leq 2b^d m_h(E)$. Then $\mu(E_2) \leq \sum \mu(Q_k) < \varepsilon \sum h(\operatorname{diam} Q_k) \leq 2b^d \varepsilon m_h(E) \to 0$ $\blacksquare$

Remark. Instead of taking $\lim_{\substack{n \to \infty \\ Q_n \cap E \neq \emptyset}} \frac{\mu(Q_n(x))}{h(b^{-n})}$, could be $\lim_{\substack{A \in \mathcal{E},\ x \in A \\ |A| \to 0}} \frac{\mu(A)}{h(f(|A|))}$ for any family $\mathcal{E}$ and the map f with the property that $\exists g(t)$, $\lim_{t \to 0} g(t) = 0$ $\forall \varepsilon > 0$ $\exists$ cover of E by elements of $\mathcal{E}$ with $|E| < \varepsilon$ and $\sum h(f(|A|)) \leq g(m_h(E))$. In particular, one can take $\mathcal{E} = ((B(x,r))_{r>0,\ x \in E}$.

Thm. Let E be compact, $m_\alpha(E) < \infty$ ($\alpha > 0$), $m_{1/\log t}(E) < \infty$ ($\alpha = 0$). Then $Cap_\alpha(E) = 0$.

Corollary. For $\alpha > H\dim E$, $Cap_\alpha E = 0$. $\alpha < H\dim E$ (by previous Thm). $Cap_\alpha(E) > 0$. $H\dim E = \inf\{\alpha : Cap_\alpha E = 0\} = \sup\{\alpha : Cap_\alpha(E) > 0\}$.

Proof. Let μ – probability measure on E.

Let $h_\alpha(t) = \begin{cases} t^\alpha, & \alpha > 0 \\ 1/\log t, & \alpha = 0 \end{cases}$

E_1 – singularity set from previous lemma with $h = K_\alpha$.

$x \notin E_1 \iff \exists Q_{n_i}(x), \eta(x) > 0 : \mu(Q_{n_i}(x)) > \eta(x) h_\alpha(b^{-n_i})$.

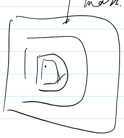
Let us prove that (x) $U_\alpha^\mu(x) = \infty$. Then, since $\mu(E_1) = 0$, $I_\alpha(\mu) = \int U_\alpha^\mu(x)\, d\mu(x) = \infty$.

If $\mu(x) > 0$, (x) is obvious. If not, select $\varepsilon_i > 0$ for $Q_i := \{y \in Q_{n_i}(x) : \|y - x\| > \varepsilon_i\}$, $\mu(Q_i) > \frac{1}{2}\eta(x) K_\alpha(b^{-n_i})\}$

Taking subsequence, we can assume that $Q_i \cap Q_k = \emptyset$, $j \neq k$.

So $U_\alpha^\mu(x) \geq \sum_i \int_{Q_i} K_\alpha(x-y)\, d\mu(y)$

Large mass.



If $\alpha > 0$, $\int_{Q_i} K_\alpha(x-y)\, d\mu(y) \geq (\sqrt{d}\, b^{-n_i})^{-\alpha} \mu(Q_i) \geq d^{-\alpha/2} b^{n_i \alpha} (b^{-n_i})^{\alpha} \frac{1}{2}\eta(x) \geq \frac{d^{-\alpha/2}}{2}\eta(x)$. So $\sum_i = \infty$.

Similarly, for $\alpha = 0$.

Let us see the precision of this for Cantor sets with side sequence ℓ_n in $\mathbb{R}^d$.

Thm. Let E be (ℓ_n) – Cantor set in $\mathbb{R}^d$. Then

$$Cap_\alpha E > 0 \iff \sum 2^{-nd} K_\alpha(\ell_n) < \infty.$$

Pf Let $\sum 2^{-nd} K_\alpha(\ell_n) < \infty$. Let us prove $(\Leftarrow)$ for $\alpha = 0$. Define $h(t) = \begin{cases} 2^{-n}, & t = \ell_n \\ \text{linear } \ell_n < t < \ell_{n-1} \end{cases}$. Then

Motivic ≥ 0 $\Downarrow$ Cap > 0

$$\int_0 \frac{h(t)}{t}\, dt \leq \sum \int_{\ell_{n-1}}^{\ell_n} \frac{2^{-n+1}}{t}\, dt = \sum 2^{-n+1} \log \frac{\ell_n}{\ell_{n-1}} \leq 2\sum 2^{-n} \log \frac{1}{\ell} < \infty.$$

Metric $\neq 0$ $\Downarrow$ Cap > 0

$\alpha = 0$. Define $h(t) = \begin{cases} 2^{-n}, & t = \ell_n \\ \text{linear} & \ell_n < t < \ell_{n-1} \end{cases}$. Then

$$\int_0 \frac{h(t)}{t}\,dt \leq \sum \int_{\ell_{n-1}}^{\ell_n} \frac{2^{-n+1}}{t}\,dt = \sum 2^{-n+1} \log \frac{\ell_n}{\ell_{n-1}} \leq 2 \sum 2^{-n} \log \frac{1}{\ell_n} < \infty.$$

and the standard dyadic measure is h-smooth, so $H_h(E) > 0$ for some h with $\int_0 \frac{h(t)}{t^{1+\alpha}}\,dt < \infty \Rightarrow \mathrm{Cap}_\alpha(E) > 0$.

On the other hand, for any measure μ, $\operatorname{supp}\mu \subset E$, and $I^\alpha(\mu) < \infty$, we have:

$$I^\alpha(\mu) = \int d\mu(y) \int K_\alpha(x-y)\,d\mu(x) = \int d\mu(y) \left(\int_{r=0}^{\infty} K_\alpha(r)\, d\mu(B(y,r)) \right)$$

$$\overset{\text{parts}}{\approx} \int d\mu(y) \left(\int \mu(B(y,r)) \frac{dr}{r^{1+\alpha}} \right) \geq$$

$$\sum_{n=0} \int_{\ell_n}^{\ell_{n-1}} \frac{dr}{r^{1+\alpha}} \underbrace{\int \mu(B(y,\ell_n))\,d\mu(y)}_{\text{there are essentially } 2^{nd} \text{ disks like that}} \geq \sum_{n=0}^{\infty} \left(K_\alpha(\ell_n) \cdot \sum_{k=1}^{2^{nd}} \mu(Q_k^n)^2 \right) \geq$$

By Cauchy, $\sum_{k=1}^{2^{nd}} \mu(Q_k^n)^2 \geq 2^{-nd} \underbrace{(\mu(E))^2}_{=2^{-nd}} \left(\frac{\sum a_i^2}{n} \geq \left(\frac{\sum a_i}{n} \right)^2 \right)$.

Thus $\infty > I^\alpha(\mu) \gtrsim \sum_{n=0}^{\infty} K_\alpha(\ell_n) 2^{-nd}$. Thus

$\mathrm{Cap}_\alpha(E) > 0 \Rightarrow \sum K_\alpha(\ell_n) 2^{-nd} < \infty$ ∎